

ON GAUGE THEORIES OF MASS

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ABSTRACT. The classical Einstein-Hilbert action in general relativity extends naturally to a blow-up (in the sense of algebraic geometry) of the usual space of pseudo-Riemannian metrics; this presents the metric tensor g_{ik} as a kind of Goldstone boson associated to the real scalar field defined by its determinant. This seems to be quite compatible with the Higgs mechanism in the standard model of particle physics.

0.1 Gauge theories of mass can be traced back to Weyl's early work on conformal geometry [10 §1, 15 §16], in light of de Broglie's quantum-mechanical relation

$$mc^2 = \hbar\nu$$

between energy and frequency; since then, work in particle physics culminating in the standard model has made the subject predictive. This note argues that relations between physical mass and (Lorentz-Minkowski) geometry are best understood by thinking of the usual pseudometric tensor as the product of a 'dimensionless' unimodular tensor (roughly $|\det g|^{-1/n} g_{ik}|$) with a suitable power of a density $\gamma \sim |\det g|^{-1/2n}$ (a section of a real line bundle which transforms like an inverse length). This enlarges the class of possible states by allowing $|\det g| \rightarrow 0$ or ∞ , provided the conformal behavior is otherwise good.

Models of this sort have a long history in physics, but related recent developments in Riemannian geometry seem not to have had much impact there. In particular, Yamabe's nonlinear elliptic eigenvalue equation [11, 16]

$$\left[-\Delta + \frac{1}{4} \frac{n-2}{n-1} R(g) - \Lambda |u|^{4/(n-2)} \right] u = 0$$

whose solutions $u > 0$ define conformal deformations $\bar{g} := u^{4/(n-2)} g$ of the metric g with scalar curvature

$$R(\bar{g}) = 4 \frac{n-1}{n-2} \Lambda = \text{constant}$$

is (apart from the change from Riemannian to Minkowski signature) identical with a (super-renormalizable, if $n = 3, 4$ or 6) conformally invariant nonlinear wave equation with its own distinguished literature (cf. eg [3, 9, 13 §15.2]).

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0.2 Aside from an example in §4.2, there is little in the present account of this material that is new, but the literature of these questions is old, broad, and perhaps confusing. My sense is that the subject is simpler than one might think; but there are nontrivial conceptual issues at stake, which are hard to present clearly.

I’ve tried to simplify things by keeping the language, and especially the physics, as classical as possible: in particular, though there is some discussion of the Higgs mechanism in §2.2, it occurs here only as a **classical** field theory; delicate quantum-mechanical issues involved in its renormalization are completely ignored. However, there are some mathematical subtleties: the nonlinear eigenvalue problems considered here may admit weak (ie moderately non-smooth) solutions, which correspond to phase transitions in some physical models. The final example suggests the possibility that a change in conformal structure in the interior Schwarzschild region might correspond to a change in the physical vacuum state there.

The paper is organized as follows: §1 summarizes some background, eg a blow-up of spaces of quadratic forms at 0, basic facts about densities on manifolds, and associated moduli spaces of geometric data. The next section uses this formalism to identify the Einstein-Hilbert action of classical relativity with a version of Yamabe’s conformally invariant functional. §3 notes some consequences of the Sobolev embedding $L_1^2 \subset L_0^{2n/(n-2)}$ which seem to fit with old ideas from physics about Mach’s principle, and §4 discusses some classical examples in this framework.

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§1 Desingularizing quadratic forms at 0

1.1 A nondegenerate symmetric bilinear form q of signature k on a finite-dimensional real vector space V has a group $O(q) \subset \mathrm{Gl}(V)$ of linear isomorphisms. If n is the dimension of V , the homogeneous space $\mathrm{Gl}(V)/O(q)$ is isomorphic to the space $Q_n^k(V)$ of nondegenerate quadratic forms of signature k on V . This is an open cone, and its quotient

$$\mathbb{G}_m \backslash \mathrm{Gl}(V)/O(q) : \cong |Q|_n^k(V)$$

by the center of $\mathrm{Gl}(V)$ is a space of ‘unimodular’ quadratic forms of signature k , isomorphic to the (contractible) space of real $n \times n$ symmetric matrices whose eigenvalue configuration λ_* consists of $n_{\pm} = \frac{1}{2}(n \pm k)$ positive (resp. negative) real numbers satisfying

$$(-1)^{(n-k)/2} \prod \lambda_* = 1.$$

For any real s , let $\mathbb{R}(s)$ be the one-dimensional real representation of $\mathrm{Gl}_n(\mathbb{R})$ defined by

$$\mathrm{sgn}(\det) \cdot |\det|^{s/n} : \mathrm{Gl}_n(\mathbb{R}) \rightarrow \mathbb{G}_m ;$$

it has a complex analog when $s \in \mathbb{C}$.

The quotient \tilde{Q}_n^* of $Q_n^* \times \mathbb{R}(\frac{1}{2}n - 1)$ defined by the action

$$\rho \cdot (q, r) \mapsto (|\rho|^{4/(n-2)} q, \rho^{-1} r)$$

of $\rho \in \mathbb{R}^\times$ is a blowup

$$q \mapsto [q, 1] : Q_n^* \rightarrow \tilde{Q}_n^* \cong |Q|_n^* \times \mathbb{R}$$

(the right-hand map sends $[q, r]$ to $(|\det q|^{-1/n} \cdot q, |\det q|^{(n-2)/4n} r)$) which replaces the cone of quadratic forms by the corresponding cylinder.

1.2 For simplicity I'll assume M compact closed, with principal frame bundle $\mathrm{Gl}(T_M)$, and I will write $\underline{\mathbb{R}}(s)$ for the real line bundle

$$\mathbb{R}(s) \times_{\mathrm{Gl}_n(\mathbb{R})} \mathrm{Gl}(T_M) \rightarrow M$$

with fiber $\mathbb{R}(s)$. There is a canonical isomorphism

$$\Lambda^n T_M^* \cong \underline{\mathbb{R}}(n)$$

between n -forms on M and sections of $\underline{\mathbb{R}}(n)$, and hence a canonical Lebesgue functional on the space $\Gamma \underline{\mathbb{R}}(n)$ of sections over M ; sections of $\underline{\mathbb{R}}(0)$, on the other hand, are ordinary real-valued functions. When $n \geq s \geq 0$, let $\mathbb{L}(s)$ denote the real Banach space of (equivalence classes of) sections of $\underline{\mathbb{R}}(s)$, completed with respect to the norm

$$\|\phi\|_s := \left[\int_M |\phi|^{\otimes n/s} \right]^{s/n}$$

(or by the essential supremum when $s = 0$). Hölder's inequalities define a canonical isomorphism

$$\mathbb{L}(s)^* \cong \mathbb{L}(n - s)$$

and continuous pairings

$$\mathbb{L}(s) \otimes_{\mathbb{R}} \mathbb{L}(t) \rightarrow \mathbb{L}(s + t)$$

when $0 \leq s, t, s + t \leq n$. $\Gamma \underline{\mathbb{R}}(s) \subset \mathbb{L}(s)$ will denote the subspace of smooth sections.

1.3 A pseudometric of signature k on M is a smooth section of the bundle

$$Q_n^k(M) := \mathrm{Gl}(T_M) \times_{\mathrm{Gl}_n(\mathbb{R})} Q_n^k \rightarrow M ;$$

it defines a smooth section

$$*_g^{s/n}(1) := \mathrm{sgn}(\deg g) \cdot |\det g|^{s/2n} \in \mathbb{L}(s)$$

of $\underline{\mathbb{R}}(s)$, as well as an isometry

$$\phi \mapsto *_g^{-s/n} \phi = f : \mathbb{L}(s) \rightarrow L^{n/s}(M, d\mathrm{vol}_g) .$$

A pseudometric g and a density $\phi \in \mathbb{L}(\frac{1}{2}n - 1)$ define, at each point $x \in M$, an element $[g(x), \phi(x)] \in \tilde{Q}_n^*(T_{M,x})$ and thus a map

$$\Gamma Q_n^* \times \Gamma \mathbb{R}(\frac{1}{2}n - 1) \rightarrow \Gamma \tilde{Q}_n^*$$

which sends the pair (g, ϕ) to a section of the bundle of blown-up pseudometrics. However it is really the ray defined by ϕ that is significant, and it will be useful to assume that

$$\phi \in \Gamma' \mathbb{R}(\frac{1}{2}n - 1) := \Gamma \mathbb{R}(\frac{1}{2}n - 1) - \{0\} .$$

This constructs a kind of coarse moduli space of generalized pseudometrics, which has a finer analog: an element u of the group $\Gamma \mathbb{R}(0)^\times$ acts on (g, ϕ) , sending it to $(|u|^{4/(n-2)}g, u^{-1}\phi)$, and the map to $\Gamma \tilde{Q}_n^*$ factors through the quotient of this action.

§2 A conformally invariant model

“There is nothing in the world bigger than the tip of an autumn hair, and Mount T'ai is little.”

Chuang Tzu, **Discussion on making all things equal**, tr.
Burton Watson

2.1.1 Proposition: The diagram

$$\begin{array}{ccccc} \Gamma Q_n^* \times \Gamma' \mathbb{R}(\frac{1}{2}n - 1) & \longrightarrow & \Gamma Q_n^* \times_{\Gamma \mathbb{R}(0)^\times} \Gamma' \mathbb{R}(\frac{1}{2}n - 1) & \longrightarrow & \Gamma \tilde{Q}_n^* \\ \uparrow 1_Q \times \gamma & & \searrow Y & & \vdots ? \\ \Gamma Q_n^* & \xrightarrow{\hbar^{-1}\mathcal{E}} & & & \mathbb{R} \end{array}$$

commutes; where

$$\mathcal{E} = \frac{1}{2}\kappa^{-1} \int_M R(g) d\text{vol}_g$$

is the Einstein-Hilbert action functional (with $\kappa = 8\pi G$),

$$Y[g, \phi] = \frac{1}{2} \int_M \left[|df|_g^2 + \frac{1}{4} \frac{n-2}{n-1} R(g) f^2 \right] d\text{vol}_g$$

is Yamabe's conformally invariant quadratic form¹(with f as in §1.3), and

$$\gamma := *_g^{(n-2)/2n} \left(\left(\frac{n-2}{n-1} G h \right)^{-1/2} \right) .$$

Proof: This is an absolute triviality (except for the assertion that Y is conformally invariant, which is now classical [11, 16]). Note however that the dotted arrow is not asserted to exist. \square

¹Here Y is normalized as if it were the Lagrangian in a Feynman measure of the form $\exp(-iL_{\text{matter}}(\psi)/\hbar) D\psi$.

2.1.2 This reformulates the Einstein-Hilbert action as the Lagrangian for a conformally invariant physical theory involving a unimodular pseudometric (ie a section of the bundle Q_4^{-2}) and a real spin-zero gauge field γ , defined locally by measurements of Newton's constant, with symmetry broken on the locus where $\gamma \rightarrow 0$ (or ∞ , if we allow noncompact M). Away from this set, conformal invariance allows us to assume that (the function corresponding to the density) γ is **constant**, ie roughly the Planck frequency

$$(\tfrac{2}{3}Gh)^{-1/2} \sim 90.7 \times 10^{35} \text{ MHz} .$$

This, after all, is what a gauge theory does; we understand γ to be constant because it **defines** the local mass scale.

At first sight Y looks like the Lagrangian for a real scalar boson, moving in a potential field of the form $R(g)$ [9]; but requiring that ϕ not be identically zero can be interpreted as the introduction of a self-interaction term. The Sobolev embedding theorem says that (on a compact smooth n -manifold) the space L_s^p of functions with s derivatives in L^p embeds in L_t^q iff $t - n/q \leq s - n/p$: in particular, $L_1^2 \subset L_0^{2n/(n-2)}$ is just on the edge of continuity. Requiring that γ have fixed norm as a $(\frac{1}{2}n - 1)$ -density, ie that its Lebesgue $2n/(n-2)$ -norm be finite, is equivalent to adding a Lagrange multiplier term of the form

$$\Lambda(|\gamma|^{2n/(n-2)} - 1) ,$$

to the Lagrangian; which, when $n = 4$, is equivalent to allowing the ‘dilaton’ γ a quartic (super-renormalizable) self-interaction.

2.1.3 From this point of view, the ‘graviton’ (ie, the field represented by the rank two symmetric tensor g_{ik}) is a Goldstone boson associated to γ : if we ‘decouple’ the metric from its determinant by writing

$$g_{ik} := \phi^{4/(n-2)} \bar{g}_{ik}$$

with

$$\phi = (|\det g|^{1/2})^{(n-2)/2n} \in \Gamma \underline{\mathbb{R}}(\tfrac{1}{2}n - 1)$$

(so $|\det \bar{g}|^{1/2} = 1$), then Yamabe's equation

$$\int *_g R(g) = \int \left[\phi^2 R(\bar{g}) + 4 \frac{n-1}{n-2} |d\phi|_{\bar{g}}^2 \right] d^n x ,$$

is completely analogous to Goldstone's identity

$$|d(e^{i\theta} \chi)|_g^2 = \chi^2 |d\theta|_g^2 + |d\chi|_g^2 .$$

The opposite interpretation – that the dilaton is a Goldstone boson associated to the metric – is more usual in physics [12]. The interpretation here is that the boson associated to the conformally invariant wave equation is more significant locally, while (perturbations of) the Lorentz-Minkowski metric, though fundamental for geometry, become important only at quantum-mechanically vast distances.

In fact such issues go back to the earliest days of the subject. Weyl observed [15 §28] that the Einstein-Hilbert action can be written as a quadratic functional

$$S(\dot{g}) := \int g^{ik} [\Gamma_{ts}^s \Gamma_{ik}^t - \Gamma_{it}^s \Gamma_{sk}^t] d\text{vol}_g$$

in the first derivatives of g , analogous to the left-hand side of Goldstone's identity.

2.2.1 It is well-known, but perhaps quite remarkable, that the standard model of particle physics is very close to conformally invariant; it is the usual coupling to gravitation which breaks the symmetry [4]. That model involves a principal bundle $P \rightarrow M$ with compact semisimple structure group

$$G = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$$

as fiber; it postulates complex vector bundles associated to two representations of G , whose sections are called Higgs and fermion fields. I'll leave the details of these representations unspecified (see [5]): for our purposes the significant fact is that these are both bundles of $(\frac{1}{2}n - 1)$ -densities.

The Lagrangian density of the standard model is a functional on a configuration space of fields (A, Φ, Ψ) , equal to the sum of

- a Yang-Mills term $*_g |F_A|^2$ defined by the curvature F_A of a connection one-form A on P ,
- a Dirac term $\Psi^\dagger \cdot i \not{\partial}_A \Psi = L(A, \Psi)$, and
- a Higgs term $L(A, \Phi)$ of the form $|d_A \Phi|_g^2 + P(\Phi)$, with the latter term a polynomial in Φ , eg something like

$$P(\Phi) = (|\Phi|^2 - \lambda^2 \phi^2)^2$$

(with a **dimensionless** coupling constant λ).

The Yang-Mills term is conformally invariant in dimension four, and if we rewrite the auxiliary fields

$$\Phi := *_g^{(n-2)/2n} \Phi_0, \quad \Psi := *_g^{(n-2)/2n} \Psi_0$$

in terms of fields (Φ_0, Ψ_0) of conformal weight zero, then the rescaling $g \mapsto u^{4/(n-2)} g$ sends

$$L(A, \Psi_0) \mapsto L(\tilde{A}, \Psi_0), \quad L(A, \Phi_0) \mapsto L(\tilde{A}, \Phi_0)$$

with $\tilde{A} := A + u^{-1} du$.

2.2.2 Sections \mathfrak{g} of the bundle

$$G^{\text{ad}} \times_G P \rightarrow M$$

(where G^{ad} is defined by the conjugation action of G on itself) form a group $\mathcal{G}(G)$ of gauge transformations, which act on the space \mathcal{A} of connections on

P by $(\mathbf{g}, A) \mapsto A - d\mathbf{g} \cdot \mathbf{g}^{-1}$. The standard model Lagrangian is thus a function on the quotient

$$(\text{Higgs} \times \text{Spinors}) \times_{\mathcal{G}(G)} \mathcal{A}$$

(with $\mathbf{g}(\Phi, \Psi) = (\mathbf{g}\Phi, \mathbf{g}\Psi)$ on the Higgs and fermion fields).

The analogous symmetry group for the geometric sector is the group \mathbb{D} of (orientation and spin-structure-preserving) diffeomorphisms of M . This acts on $\mathcal{G}(G)$, and the moduli space of states for the standard model coupled to the usual version of general relativity is a bundle

$$(\text{Higgs} \times \text{Spinors}) \times_{\mathcal{G}(G)} \mathcal{A} \rightarrow (\cdots) \rightarrow \Gamma Q_n^* / \mathbb{D}.$$

2.2.3 The multiplicative group $\Gamma\mathbb{R}(0)^\times$ of nowhere-vanishing real-valued functions on M [§1.3] can be equally well regarded as a group $\mathcal{G}(\mathbb{G}_m)$ of gauge transformations associated to a principal bundle with structure group the noncompact torus $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$. If we interpret the fields of the standard model as densities as above, then its moduli space of field configurations can be presented as the quotient

$$(\text{Higgs} \times \text{Spinors}) \times_{\mathcal{G}(\tilde{G})} \tilde{\mathcal{A}}$$

with

$$\tilde{\mathcal{A}} := \mathcal{A} \times d\Omega^0(M), \quad \mathcal{G}(\tilde{G}) := \mathcal{G}(G \times \mathbb{G}_m) \cong \mathcal{G}(G) \times \mathcal{G}(\mathbb{G}_m)$$

under the action

$$(\mathbf{g}, u) \cdot (A, \Psi, \Phi) := (A - u^{-1}du - d\mathbf{g} \cdot \mathbf{g}^{-1}, u^{-1}\mathbf{g}\Psi, u^{-1}\mathbf{g}\Phi).$$

The Lagrangian of the standard model coupled to gravitation then extends, as in §2.1, to a function on the quotient of the space

$$((\text{Higgs} \times \text{Spinors}) \times \tilde{\mathcal{A}}) \times_{\mathcal{G}(\mathbb{G}_m)} (\Gamma Q_n^* \times \Gamma' \mathbb{R}(\tfrac{1}{2}n - 1))$$

of fields by the gauge group $\mathcal{G}(\tilde{G}) \rtimes \mathbb{D}$. This defines a conformally invariant version of the standard model coupled to gravity, whose solutions are classical off the singular locus $\gamma^{-1}(0)$.

2.2.4 This suggests several possible directions of extension, which I hope to discuss later:

- the Connes-Chamseddine-Lott noncommutative version of the standard model [4]
- supersymmetric (eg minimal) extensions of the standard model,
- Connes-Kreimer-Marcolli renormalization of the conformally invariant 4D ϕ^4 model (which involves treating mass as a gauge field) [1]; and
- questions of classical analysis: I've evaded certain issues by assuming the underlying space-time manifold to be compact, and by working entirely with smooth sections. However, §4.2 below suggests the interest of weak (eg L_1^2) solutions to these equations.

§3 Inferences from scale invariance

“ To show the fly the way out of the fly-bottle ... ”

Ludwig Wittengstein, **Philosophical Investigations**

Invariance under rescaling by a **constant** factor $\rho > 0$ may help illuminate some basic issues of physical interpretation:

$$R(\rho^{4/(n-2)}g) = \rho^{-4/(n-2)}R(g) ,$$

so

$$\|R(g)\|_{L^{n/2}(g)} = \left[\int_M |R(g)|^{n/2} d\text{vol}_g \right]^{2/n}$$

is scale-invariant, i.e. a ‘pure number’. In view of the extensive literature concerned with anomalously large or small cosmological numbers, it seems remarkable that this L^2 -norm (when $n = 4$) seems to be well-behaved in the standard astrophysical models: the current bound for the cosmological constant is roughly

$$|\Lambda| \leq 10^{-35} \text{ sec}^{-2} ,$$

while the universe is thought to be something like 4×10^{17} seconds old, suggesting that

$$\|R\|_{L^2} \sim O(1)$$

(consistent with the hypothesis, plausible on other grounds, that $R = 0$).

More generally, Hölder’s inequalities imply that for a smooth real-valued function f on M we have

$$\|R(g)f^2\|_{L^1(g)} \leq \|R(g)\|_{L^{n/2}(g)} \cdot \|f\|_{L^{2n/(n-2)}(g)}^2 .$$

The norm appearing on the right rescales like a length:

$$\|f\|_{L^{2n/(n-2)}(\rho^{4/(n-2)}g)} = \rho \|f\|_{L^{2n/(n-2)}(g)} ,$$

which suggests regarding it as an estimate of the ‘radius’ of M (i.e. the n th root of its volume), measured in units defined by f . On the other hand, interpreting f as the inverse Planck length suggests regarding $R(g)f^2$ as an analog of the stress-energy scalar

$$T \sim \kappa^{-1}R(g) .$$

In fact

$$\|R(g)f^2\|_{L^1(\rho^{4/(n-2)}g)} = \rho^2 \|R(g)f^2\|_{L^1(g)}$$

scales like $\|T\|_{L^1(g)} \sim \text{Energy}^2 \cdot \text{Hypervolume}$, cf. [8 §3.3]. The inequality

$$\|R\|_{n/2} \geq \|Rf^2\|_1 \cdot \|f\|_{2n/(n-2)}^{-2}$$

can therefore be interpreted as bound of the form

$$\text{Const} \geq \frac{\text{Mass}}{\text{Radius}} ;$$

conceivably this lies behind the ‘numerical coincidences’ which physicists have interpreted as evidence for some version [3] of Mach’s principle.

§4 Two examples

“The further in you go, the bigger it gets,” said Hannah Noon.’ John Crowley, **Little, Big**

I’ll close with an attempt to show that this formalism is not completely without content. The first example below is quite widely known [cf. eg. [9 §7]], but the second is more speculative.

4.1.1 The positively curved Friedman pseudometric on $\mathbb{R}^4 \cong \mathbf{F}_+$ is defined by

$$g = \begin{bmatrix} 1 & 0 \\ 0 & -R^2 A^2 \mathbf{1} \end{bmatrix},$$

where $A = (1 + \frac{1}{4}r^2)^{-1}$ and $r^2 = x_1^2 + x_2^2 + x_3^2$. The expansion factor² $R(t)$ ($t = x_0$) satisfies the differential equation

$$\left(\frac{dR}{dt}\right)^2 = \frac{R_0 - R}{R}, \quad R(0) = 0;$$

it increases to a maximum R_0 at time t_0 , and then decreases to zero. The conformally equivalent pseudometric

$$R(t)^{-2} dt^2 - A^2 \sum_{1 \leq k \leq 3} dx_k^2$$

on the stereographic completion $\mathbb{R} \times S^3$ is isometric to a product manifold with time parameter $d\tau = R(t)^{-1} dt$, i.e. such that

$$\left(\frac{dR}{d\tau}\right)^2 = (R_0 - R)R,$$

which is satisfied by $R(\tau) = R_0 \sin^2(\tau/2)$. In other words, the spherical Friedman model is conformally equivalent to the universal cover of the compact manifold $S^1 \times S^3$ with the standard product Lorentz metric, endowed with a dilaton field (proportional to $\sin^2(\tau/2)$) which vanishes smoothly at singularities (big bangs) recurring with period 2π . [This corresponds to identifying the upper and lower edges in the Penrose diagram displayed in Fig. 21(ii) of [8 §5.3].]

4.1.2 The (hyper)volume of this model (i.e. of one cycle of the closed Friedman universe) seems not to be well known: it equals

$$\int_{\mathbf{F}_+} *_g 1 = 2 \cdot \text{Vol}(S^3) \cdot \int_0^{t_0} R(t)^3 dt.$$

²I’m following the notation of [7 p. 117], but with R denoting the Robertson-Walker factor.

If $w := R/R_0$ then

$$R_0 w' = (w^{-1} - 1)^{-1/2},$$

so

$$dt = R_0(1 - w)^{-1/2} w^{-1/2} dw$$

and hence

$$\int_0^{t_0} R(t)^3 dt = R_0^4 \int_0^1 w^{5/2} (1 - w)^{-1/2} dw.$$

Substituting $w = \sin^2 \theta$ yields $2 \int_0^{\pi/2} \sin^6 \theta d\theta$ for the right-hand integral; this equals

$$\frac{1}{96} [60\theta - 45 \sin 2\theta + 9 \sin 4\theta - \sin 6\theta] \Big|_0^{\pi/2} = \frac{5}{16} \pi,$$

yielding (tip o'the hat to Archimedes) the value

$$2 \cdot 2\pi^2 \cdot \frac{5}{16} \pi \cdot R_0^4 = \frac{5}{4} \pi^3 \cdot R_0^4$$

for the volume of one Friedman \mathbb{A} on.

4.2.1 The Schwarzschild metric

$$d\tau^2 = (1 - 2mr^{-1})dt^2 - (1 - 2mr^{-1})^{-1}dr^2 - r^2 d\sigma^2$$

is usually defined on the spacetime manifold $\mathbb{R} \times \mathbb{R}^3 - 0$;

$$d\sigma^2 = d\phi^2 + \sin^2 \phi d\theta^2$$

is the standard metric on the two-sphere. More generally, the expression

$$d\tau^2 = q(r)r^{-2}dt^2 - q(r)^{-1}r^2dr^2 - r^2d\sigma^2$$

with

$$q(r) = r^2 - 2mr + e^2 = (r - m - D)(r - m + D)$$

defines the Reissner-Nordstrom metric; the discriminant $D^2 = m^2 - e^2$ will be assumed positive here. Both of these examples have vanishing scalar curvature, but if $e > 0$ Reissner-Nordstrom space is not Ricci-flat.

In null coordinates v, w such that $t = \frac{1}{2}(v + w)$ and

$$\frac{1}{2}(v - w) = \int q(r)^{-1} r^2 dr$$

the expression above defines the pseudometric

$$d\tau^2 = qr^{-2}dvdw - r^2d\sigma^2$$

with associated volume element

$$|g|^{\frac{1}{2}} dv \wedge dw \wedge d\theta \wedge d\phi = \frac{1}{2} q \Omega dv \wedge dw = \Omega r^2 dr \wedge dt,$$

where $\Omega = \sin \phi d\theta \wedge d\phi$ is the volume element for the two-sphere. The topology of the maximal analytic extension of such a pseudometric is complicated; see, for example, [8 §5.5]. Here we will be concerned mostly with the region of type II, in the terminology used there; this corresponds to the condition $r \in (m - D, m + D)$.

4.2.2 In these coordinates the Laplace-Beltrami operator

$$\Delta f = |g|^{-\frac{1}{2}} [|g|^{\frac{1}{2}} g^{ik} f_{,k}]_{,i}$$

takes the form

$$\Delta f = 2q^{-1} [(r^2 f_{,v})_{,w} + (r^2 f_{,w})_{,v}]$$

for a function $f = f(v, w)$ independent of θ and ϕ . If f is independent of t as well, and we use primes to denote differentiation with respect to r , then $v' = q^{-1}r^2$ and

$$\Delta f = 2q^{-1} (r^2 f' / q^{-1} r^2)' / q^{-1} r^2 = 2r^{-2} (q f')'.$$

There is thus a two-dimensional family of elementary harmonic functions $u = u(r)$ in the interior Schwarzschild region, characterized by the condition $u' = kq^{-1}$ for some constant k . Ignoring the time parameter, these functions are all of Lebesgue class L^4 , but their derivatives are only locally of Lebesgue class L^2 . Since $u'' = -kq^{-2}q'$, any such solution has a point of inflection at $r = m$. The function defined by

$$U(r) = 1 + \frac{D}{m} \log \left| \frac{r - m - D}{r - m + D} \right|$$

if $r \in (m - D, m)$, and $U(r) = 1$ if $r \geq m$, is particularly interesting. It is continuous, but not differentiable, at $r = m$; its derivative is

$$U' = 2m^{-1} D^2 q^{-1} \eta,$$

where η denotes a unit step function at $r = m$, so

$$\Delta U = 4m^{-3} D^2 \delta$$

with δ a Dirac delta-function at $r = m$. The piecewise-differentiable tensor $\bar{g} = U^2 g$ thus has scalar curvature

$$\bar{R} = -24m^{-3} D^2 \delta$$

which vanishes almost everywhere [14].

4.2.3 The conformally deformed tensor \bar{g} will not be Ricci-flat, even in the Schwarzschild case; a straightforward calculation [2 §6.3] shows that for a general conformal deformation $\bar{g} := u^{4/n-2} g$,

$$\bar{R}_k^i := R_k^i(\bar{g}) = u^{-4/n-2} R_k^i(g) + u^{-2n/n-2} P_k^i$$

with

$$P_k^i = -2u[\nabla_k^i u + (n-2)^{-1} \delta_k^i \Delta u] + 2(n-2)^{-1} [n \nabla^i u \nabla_k u - \delta_k^i |\nabla u|^2]$$

where ∇_i signifies covariant differentiation [which on scalars is to be interpreted as ordinary differentiation]. If we assume that $n = 4$ and $u' = kq^{-1}$ as above, then the resulting tensor is diagonal in Schwarzschild coordinates; its entries have invariant significance, as the eigenvalues of P_k^i considered as an endomorphism of the tangent space. Displaying these diagonal components as vectors, we find that P_k^i equals

$$kr^{-2} q^{-1} q' u [1, -1, 0, 0] + 2kr^{-3} u [-1, -1, 1, 1] + k^2 r^{-2} q^{-1} [1, -3, 1, 1];$$

if $u' = kq^{-1}\eta$ is cut off at $r = m$, there is an additional term of the form $-kr^{-2}\delta[0, +1, 0, 0]$. Assuming $u = 1$ at its inflection point, $k = 2m^{-1}D^2$ is the unique value for which the determinant of P_k^i vanishes at $r = m$; this characterizes the harmonic function U .

It is similarly straightforward to show that (as in the Schwarzschild case), the equation of a radial timelike geodesic in the metric $\bar{g} = U^2g$ becomes

$$U^4\dot{r}^2 = b^2 + (2mr^{-1} - 1)U^2,$$

where the dot denotes differentiation with respect to proper time, and b is a constant of integration. When r is small,

$$d\tau \sim (2m)^{-1/2}r^{1/2}U \, dr$$

is integrable; as in the classical case, such a geodesic reaches the origin in finite proper time.

4.2.4 This suggests the interest of solutions of the equation

$$\Delta u + \Lambda u^3 = 0$$

with $\Lambda \neq 0$. Taking $e = 0$ and $m = \frac{1}{2}$ for simplicity, and assuming as above that u depends only on the radial coordinate, this becomes

$$2r^{-2}(r(r-1)u')' + \Lambda u^3 = 0,$$

which bears some (superficial?) resemblance to the Lane-Emden equation of astrophysics [6 Ch. IV].

By Cauchy-Kowalevskaya, the equation above has a unique solution analytic near 0 of the form

$$3(2\Lambda)^{-1/2}v(r) := 3(2\Lambda)^{-1/2}r^{-3/2}(1 + \sum_{k>0} w_k r^k),$$

where

$$(r(r-1)v')' + \frac{9}{4}r^2v^3 = 0.$$

In terms of $v(r) = r^{-3/2}w(r)$, the equation above becomes

$$4r^2(r-1)w'' + 4r(2-r)w' + 3(r-3)w + 9w^3 = 0,$$

which is satisfied by

$$w = 1 - \frac{3}{26}r - \frac{165}{26^2}r^2 + \dots$$

Numerical computations (many thanks to S. Agarwala and Dan Christensen for invaluable help, including infinitely many corrections) suggest this series has radius of convergence **one**, and that w is nonvanishing in the interval $[0, 1)$.

The (orientation-reversing) monotonic change of variables $t = \log |r^{-1} - 1|$ maps $(0, 1)$ to $(-\infty, \infty)$. Rewriting the equation above as

$$v''(t) = \frac{9}{4} \frac{e^t}{(1+e^t)^4} v(t)^3$$

suggests that $v''(t) > 0$ for all $t \in \mathbb{R}$, and hence that the graph of v is concave upwards. Since $v(r) \rightarrow \infty$ as $r, s \rightarrow 0$, this implies the existence of a unique critical point $v'(\rho) = 0$ with $\rho \in (0, 1)$. Numerical calculations suggest

$$\tilde{v}(r) = r^{-3/2}(1-r)^{-1/2}(1 - \frac{2}{3}r)$$

as a reasonable approximation to v away from $r = 1$; it has a unique critical point at

$$\tilde{\rho} = \frac{1}{4}(7 - \sqrt{13}) \sim .85 \dots$$

Reasoning as in §4.2.2, this suggests that the function

$$V(r) = v(\rho)^{-1}v(r) \text{ if } r \in (0, \rho), = 1 \text{ otherwise,}$$

defines a conformal deformation of the interior Schwarzschild metric with a second-order phase transition at $r = \rho$, with

$$\overline{R} = 6\Lambda = 108m^2\rho^{-3}w(\rho)^2$$

when $r < \rho$, such that

$$d\overline{\tau} \sim (2m)^{-1/2} \frac{\rho^{3/2}}{w(\rho)} w(r) \cdot r^{-1} dr$$

has a logarithmic pole at $r = 0$, defining a complete metric which puts the singularity at infinity, possibly corresponding to an interesting new ground state for the interior Higgs field. The Penrose diagram for the associated conformally deformed metric glues together the vertical edges of Figure 25 in [8 §5.5], identifying horizontal pairs of parallel type I regions.

4.2.5 In terms of the coordinate t , the equation above has an asymptotic solution

$$v(t) \sim \sum_{k \geq 0} v_k t^{-k} \in \text{AC}[[t^{-1}]]$$

with coefficients in the differential Frechét algebra AC of smooth functions on \mathbb{R} with rapidly decreasing (Schwartz class) derivatives. The function

$$v_0(t) = \frac{1}{3}[1 + 8(1 + e^{-t})^{-2}]^{1/2} (= \frac{1}{3}[1 + 8(1 - r)^2]^{1/2}) \in \text{AC}$$

is an example: $v_0 \rightarrow 1$ as $t \rightarrow \infty$, $\frac{1}{3}$ as $t \rightarrow -\infty$. If we define

$$v(r) := r^{-3/2}(1-r)^{-1/2}x(r) = 4e^t \cosh^2(\frac{1}{2}t) x(t)$$

then the equation for v can be rewritten as a Duffing equation

$$x'' + \delta_1 x' + \delta_0^2 x = \frac{9}{4}x^3$$

with

$$\delta_1 = \frac{3e^t + 2 - e^{-t}}{e^t + 2 + e^{-t}}, \quad \delta_0^2 = \frac{9e^t + 2 + e^{-t}}{4(e^t + 2 + e^{-t})} \in \text{AC}.$$

This has $x_0 = \frac{2}{3}\delta_0$ (corresponding to v_0) as a kind of asymptotically stationary approximate solution.

To improve the approximation, let $x = x_0 y$ (note that δ_0 is invertible in AC). The linear operator

$$L := \delta_0^{-3}(\partial + \delta_1)\partial\delta_0 = \delta_0^{-3}(\delta_0\partial^2 + \epsilon_1\partial + \epsilon_0)$$

has coefficients

$$\epsilon_1 = 2\delta'_0 + \delta_1\delta_0, \quad \epsilon_0 = \delta''_0 + \delta_1\delta'_0$$

in the Schwartz class \mathcal{S} . It thus extends to define a map from the differential algebra $\text{AC}[[t^{-1}]]$ to $\mathcal{S}[[t^{-1}]]$.

Suppose now that $y(n) = \sum_{n \geq k \geq 0} y_k t^{-k} \in \text{AC}[[t^{-1}]]$ has been constructed, such that

$$F(y(n)) := Ly(n) + y(n) - y(n)^3 \in t^{-n-1}\text{AC}[[t^{-1}]] ;$$

we can start an induction with $y_0 = 1$, since $E_1 = t(\delta''_0 + \delta_1\delta'_0) \in \mathcal{S}$. Then

$$F(y(n) + y_{n+1}t^{-n-1}) \equiv F(y(n)) + F'(y(n)) \cdot y_{n+1}t^{-n-1} \bmod t^{-n-2}\text{AC}[[t^{-1}]]$$

with $F(y(n)) \equiv E_{n+1}t^{-n-1}$ modulo higher powers of t^{-1} , for some asymptotically constant error term E_{n+1} . The coefficient of t^{-n-1} in the term on the right above then simplifies to

$$E_{n+1} + (L - 2)y_{n+1}$$

and we can take $y_{n+1} = (2 - L)^{-1}E_{n+1}$.

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